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Superisometries of the $AdS \times S$ superspace

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Abstract: We find the superisometry of the near-horizon superspace, forming the superconformal algebra. We present here the explicit form of the transformation of the bosonic and fermionic coordinates (as well as the compensating Lorentz-type transformation) which keeps the geometry invariant. We comment on *i*) the BPS condition of the branes in AdS background and *ii*) significant simplification of superisometries of the gauge-fixed Green-Schwarz action in $AdS_5 \times S^5$ background.

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Contents

1. Introduction

It has been suggested in [1] to consider the branes as interpolating solutions between two maximally supersymmetric vacua. The first vacuum is related to the brane at infinity and is given by a flat space without forms. Flat space has the maximal amount of unbroken supersymmetry and can be understood as a flat superspace. It has a description [2] in terms of the supercoset construction G/H where G is the super Poincaré group and H is its Lorentz subgroup, see also [3] for details. The second vacuum is associated with the near-horizon geometry of the brane solutions, given by the $AdS_{p+2} \times S^{d-p-2}$ metric and a form field. It can also be understood as a superspace since the amount of unbroken supersymmetries is maximal. Recently a supercoset G/H construction [4, 5, 6] of the near-horizon superspace $AdS_{p+2} \times S^{d-p-2}$

+ form geometries was developed. Here G is the relevant (extended) superconformal group and the stability group H is the product subgroup $\mathrm{SO}(p+1,1)\times\mathrm{SO}(d-p-2)$. It was also shown that the supercoset construction of the near-horizon superspace is equivalent to a description in terms of the supergravity superspace at the fixed point where all covariant geometric superfields become covariantly constant [7, 8]. These vacua are exact as shown in [7] in the framework of supergravity. One can also understand the exactness property by the fact that the relevant supercoset construction is completely defined by the supergroup G.

We will develop here the supercoset construction of these superspaces with the purpose to find the superisometries of the $AdS\times S$ superspace. All equations will be kept in a form valid for the super Poincaré as well as for the superconformal group. This will be useful for a simple check of the consistency of our construction since in case of the flat superspace the superisometries are known. To explain our goal in the $AdS\times S$ case we recall below the superisometries of the flat superspace.

The flat superspace with coordinates $Z=\{x,\theta\}$ is defined by the vielbein superforms

$$E^a \equiv dx^\mu \delta_\mu^a + \bar\theta\gamma^a d\theta\,, \qquad E^\alpha \equiv d\theta^\alpha\,. \tag{1.1}$$

The isometries of the flat superspace are defined by the invariance of the superspace vielbeins, under the change of coordinates of the superspace $\delta Z = -\Xi(Z)$ supplemented by a compensating Lorentz transformation with the parameter $\Lambda^{ab}=\lambda_M^{\mu\nu}\delta_\mu^a\delta_\nu^b$ on the vielbeins, i.e.

$$\Delta E \equiv \mathcal{L}_\Xi E + \delta_{Lor}(\Lambda)E = 0\,. \tag{1.2}$$

One finds that the superisometries are given by

$$\Xi(Z): \qquad -\delta\theta = \epsilon + \frac{1}{4}\lambda_M^{\mu\nu}\gamma_{\mu\nu}\theta\,, \qquad -\delta x^\mu = a^\mu + \lambda_M^{\mu\nu}x_\nu + \bar\epsilon\gamma^\mu\theta\,. \tag{1.3}$$

with x,θ-independent $a^\mu,\lambda^{\mu\nu},\epsilon$, which are the parameters of a global Poincaré supersymmetry. This set of symmetries is known to be present in the classical actions of the GS-superstring[1], BST-supermembrane, $M5$-brane, all D-brane actions etc. as long as the background is a flat superspace. This symmetry follows from the fact that the worldvolume actions depend on the pull-back to the worldvolume of the vielbein forms E. The equation above states that the actions of the extended objects in the flat superspace background have manifest super Poincaré symmetry.

We would like to exhibit the analogous manifest superconformal symmetry of the actions of the extended objects placed in the $AdS\times S$ + form background. One

[1] In particular, the fact that the spectrum of states of GS string upon quantization forms the representations of the Poincaré supersymmetry can be traced back to the superisometry of the background.

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of the motivation is to use these symmetries to find the spectrum of states e.g. of the GS superstring in $AdS_5 \times S^5$ background [9, 10] which will form the representations of SU$(2,2|4)$ supergroup. For this purpose we will find the analogous transformations $\Xi(Z), \Lambda^{ab}(Z)$ for the near-horizon superspace under which the vielbein and the connection forms and therefore the classical actions are invariant.

The paper presents a new G-covariant approach to the definitions of the superisometries of the G/H supercoset space. This approach is based on the concept of the gauged [11] superalgebra $\mathbf{G}$ and is described in sec. 2. It allows us to formulate the G-covariant Killing equations in superspace and solve them to all orders in fermionic coordinates of the superspace θ in terms of the values of these Killing fields at $\theta = 0$. Our construction is independent of a coordinate choice for the bosonic space (the superspace at $\theta = 0$), which is encoded in the choice of a coset representative of the bosonic subgroup. In sec. 3 we show how to derive from the G-covariant Killing vectors in superspace the standard form of superisometries, i.e. the transformation of the coordinates of the supercoset space and the compensating transformation of the stability group H. As a warm up we apply our general formulae to the flat superspace in sec. 4 before we specialize to the particular case of interest, the $AdS_{p+2} \times S^{d-p-2}$ superspaces. We supplement our general formulae with the relevant superalgebra and the Killing fields, i.e. Killing vectors and spinors, of the bosonic space in coordinates in which the metric is a product metric. The bosonic part of the superconformal isometries was found in [12] for the case of only radial excitations on the worldvolume and extended to the most general bosonic case in [13]. Also the Killing spinors for $AdS \times S$ geometry are known [14]. We add here the relevant $\theta = 0$ expressions for the H transformations.

The $AdS \times S$ superspace has been obtained in the context of brane solutions to supergravity. In supergravity the Killing equations are the transformations that leave the supergravity solution invariant. In particular the Killing-spinor equation is the vanishing transformation of the fermionic fields at vanishing fermionic fields. In supergravity the group structure of the superspace is not manifest. We show how to derive the superisometries in supergravity superspace by interpreting the curvature and torsion constraints as Maurer Cartan equations. Particularly when the metric is not a product space, one may require this alternative description. An example is when the $AdS \times S$ metric is rewritten in cartesian coordinates, in which the metric is invariant with respect to the directions along the brane and to those transverse to the brane. The R-symmetry of the conformal group is manifest in these coordinates. The worldvolume actions of various branes are known to be much simpler in these coordinates [13, 9, 10]. However, one can not directly use the supercoset approach here since the form fields of supergravity instead of being constant ($F_{01\ldots pr} = 1$ in $AdS \times S$ case) become covariantly constant and depend on transverse coordinates as $F_{01\ldots pI} \sim \frac{y^I}{|y|}$ in cartesian case. Fortunately, our approach to the G/H supercoset space based on the concept of the gauged superalgebra $\mathbf{G}$ does not require the structure functions of the algebra to be true constants. We are able to solve for the geometry and the isometries of the supergravity superspace for all cases of the maximally supersymmetric vacua, characterized by covariantly constant superfields.

In the discussion section we suggest various possibilities to use our solution of the superisometries of the $AdS \times S$ superspace. In particular we comment on BPS states on the worldvolume. We discuss the use of isometries combined with κ-symmetry which provides the superconformal symmetry of the gauge-fixed worldvolume actions including the Green-Schwarz IIB action in $AdS_5 \times S^5$ superspace. Appendix A contains our conventions with respect to the use of super differential forms and the gauging of (soft) superalgebras. Appendix B provides some elementary useful information on the coset manifolds and superisometries.

2. A G-covariant approach to superisometries

The Killing equations defining the superisometries of the superspace can e.g. be found in [15], eq. (II.3.63). These equations generalize the bosonic equations for the isometries of the coset spaces G/H, given in [15] in eq. (I.6.72) to the supersymmetric case. In both cases the Killing equations have an H-covariant form. They are particularly difficult to solve to all orders in θ for generic cosets G/H including the $AdS \times S$ coset superspace.

2.1 G-covariant Killing equations

We will take here a different approach to the problem by presenting the G-covariant Killing equations, which will allow us to actually find the isometries of the $AdS \times S$ superspace. We will also show that our G-covariant approach is equivalent to the H-covariant one in [15].

Our starting point is a supergroup G with associated superalgebra $\mathbf{G}$. The generators of the algebra are $\mathbf{T}_\Lambda$ and satisfy

$$[\,\mathbf{T}_\Lambda, \mathbf{T}_\Sigma\,\} \equiv \mathbf{T}_\Lambda \mathbf{T}_\Sigma \mp \mathbf{T}_\Sigma \mathbf{T}_\Lambda = f_{\Lambda\Sigma}{}^{\Delta} \mathbf{T}_\Delta\,, \tag{2.1}$$

where the $+$ sign is taken if both Λ and Σ are fermionic indices and $f_{\Lambda\Sigma}{}^{\Delta}$ are the structure constants.

The standard approach to the supercoset space G/H consists of the construction of the left-invariant Cartan 1-forms L by

$$\mathcal{G}(Z)^{-1} d\mathcal{G}(Z) = L(Z) = L^\Lambda(Z)\mathbf{T}_\Lambda = dZ^M L_M{}^\Lambda(Z)\mathbf{T}_\Lambda\,, \tag{2.2}$$

where $Z^M = \{X^\mu, \theta^{\dot\alpha}\}$ are the supercoset coordinates and $\mathcal{G}(Z)$ is the coset representative. These forms satisfy the Maurer-Cartan equations

$$0 = dL - L \wedge L \equiv \left(dL^\Lambda + \frac{1}{2} L^\Delta \wedge L^\Sigma f_{\Sigma\Delta}{}^\Lambda\right) \mathbf{T}_\Lambda\,. \tag{2.3}$$

We would like to reinterpret this construction using the concept of gauging[2] the superalgebra $\mathbf{G}$. To *gauge the superalgebra* $\mathbf{G}$ means to introduce the coordinate dependent gauge field A, which is a $\mathbf{G}$-valued 1 form,

$$A(Z) \equiv A^{\Lambda}(Z)\mathbf{T}_{\Lambda} \tag{2.4}$$

and to construct a covariant derivative denoted by

$$\mathcal{D} = d - A\,. \tag{2.5}$$

We define the transformation of the gauge field $A(Z)$ with coordinate dependent $\mathbf{G}$-valued parameter $\Lambda(Z)$, such that in $\delta(\Lambda)(\mathcal{D}\Phi)^{\alpha}$, where $(\Phi)^{\alpha}$ is a covariant field, there is no derivative on the parameters $\Lambda^{\Lambda}(Z)$. This can be achieved by taking

$$\delta(\Lambda)A^{\Lambda}\mathbf{T}_{\Lambda} = d\Lambda + [\,A, \Lambda\,] = \left(d\Lambda^{\Lambda} + \Lambda^{\Delta}A^{\Lambda}f_{\Lambda\Delta}{}^{\Lambda}\right)\mathbf{T}_{\Lambda}\,. \tag{2.6}$$

One constructs the curvature F, which is a $\mathbf{G}$-valued 2 form and a covariant field transforming in the adjoint representation, by

$$\mathcal{D}^2 = -F \tag{2.7}$$

and it follows that

$$F = dA - A \wedge A = \left(dA^{\Lambda} + \frac{1}{2}A^{\Delta} \wedge A^{\Sigma}f_{\Sigma\Delta}{}^{\Lambda}\right)\mathbf{T}_{\Lambda}\,. \tag{2.8}$$

The supercoset G/H in the framework of *gauged superalgebra* has the following properties:

- The Maurer-Cartan equations associated with the supercoset construction G/H can be interpreted as the vanishing curvature of the gauged superalgebra, where the Cartan 1-forms are gauge fields A, which are *pure gauge*

$$L \equiv A\,, \qquad dL - L \wedge L = 0\,, \qquad \Longrightarrow \qquad F = dA - A \wedge A = 0\,. \tag{2.9}$$

- The gauge transformation with $\mathbf{G}$-valued parameter $\Sigma(Z)$ which keeps the gauge field *pure gauge* is defined as the covariant field $\Sigma(Z)$ with vanishing covariant derivative

$$\delta(\Sigma)A^{\Lambda}\mathbf{T}_{\Lambda} = d\Sigma + [\,A, \Sigma\,] = \left(d\Sigma^{\Lambda} + \Sigma^{\Delta}A^{\Sigma}f_{\Sigma\Delta}{}^{\Lambda}\right)\mathbf{T}_{\Lambda} = 0\,. \tag{2.10}$$

- The integrability condition for the existence of such covariant $\mathbf{G}$-valued field Σ with vanishing covariant derivative is the absence of a curvature, which is equivalent to Maurer-Cartan equations

$$d\Sigma + [\,A, \Sigma\,] = 0 \quad \Longrightarrow \quad [F, \Sigma] = 0 \quad \Longrightarrow \quad F = dA - A \wedge A = 0\,. \tag{2.11}$$

[2]The presentation below follows the notation and conventions of [11].

- A covariantly constant $\mathbf{G}$-valued field Σ being a gauge transformation (2.6), is by definition G-covariant. We will call (2.10) for the $\mathbf{G}$-valued covariantly constant field $\Sigma^\Lambda(Z)$ the *G-covariant Killing equations* and the $\Sigma^\Lambda(Z)$ are referred to as the *G-covariant Killing superfields.*

The relation of our G-covariant Killing superfields $\Sigma^\Lambda(Z)$ to the superisometries of the supercoset space G/H is as follows. The relevant equation[3] presents the change of coordinates $\delta Z^M = -\Xi^M$ and the compensating stability group H transformation with the parameter Λ^i which together keep the Cartan forms invariant:

$$\begin{aligned} 0 &= \mathcal{L}_\Xi L + d\Lambda + [\, L(Z), \Lambda\,] \\ &= \left(\mathcal{L}_\Xi L^\Lambda + d\Lambda^i \delta_i{}^\Lambda + \Lambda^i L^\Sigma f_{\Sigma i}{}^\Lambda\right) \mathbf{T}_\Lambda\,. \end{aligned} \tag{2.12}$$

One can rewrite (2.12) in G-covariant form. It turns out that this can be done by defining the local parameters

$$\Sigma^{\bar M} = \Xi^M L_M{}^{\bar M}\,, \qquad \text{and} \qquad \Sigma^i = \Lambda^i + \Xi^M L_M{}^i\,. \tag{2.13}$$

Using the Maurer Cartan equations (2.3) we find that (2.12) reduces to

$$0 = d\Sigma + [L, \Sigma] = d\Sigma^\Lambda + \Sigma^\Delta L^\Sigma f_{\Sigma\Delta}{}^\Lambda\,. \tag{2.14}$$

This is precisely the G-covariant Killing equation which we have derived in (2.11). It is now clear that if we can solve the covariant equation we will get the superisometries using (2.13). In the same way that (2.2) solves the Maurer-Cartan equations trivially, also the G-covariant Killing equation is solved in terms of the coset representative,

$$\Sigma(Z) = \mathcal{G}^{-1}(Z)\Upsilon_0\mathcal{G}(Z)\,, \tag{2.15}$$

where Υ_0 is a $\mathbf{G}$-valued *constant.* In general the Killing fields depend on all orders of θ. In the following subsection we will show how to derive the higher orders in θ in closed form in terms of the θ-independent Killing fields, under particular 'mild' assumptions about the coset representative.

2.2 The G-covariant superisometries to all orders in θ

First we will show how to solve the equations for Cartan forms to all orders in θ reproducing the result from [5] but using the compact notation of the Lie-algebra valued objects and their commutators defined above. Consider a boson-fermion split of the algebra $\mathbf{G}$,

$$\mathbf{G} = \mathbf{B} \oplus \mathbf{F}\,, \tag{2.16}$$

[3] In Appendix B we present the derivation of the superisometries for the supercoset space, following [15].

where $\mathbf{B}$ collects the bosonic generators $\mathbf{B}_A$ and $\mathbf{F}$ collects the fermionic generators $\mathbf{F}_\alpha$. We have to take into account that the algebra decomposes as

$$\begin{aligned}
[\mathbf{B},\mathbf{B}] &= \mathbf{B}\,,\\
[\mathbf{B},\mathbf{F}] &= \mathbf{F}\,,\\
\{\mathbf{F},\mathbf{F}\} &= \mathbf{B}\,.
\end{aligned} \tag{2.17}$$

We define the split of a $\mathbf{G}$-valued object A into a $\mathbf{B}$-valued and an $\mathbf{F}$-valued object according to (2.16)

$$A = A^\Lambda \mathbf{T}_\Lambda = A^{\mathbf{B}} + A^{\mathbf{F}}\,, \tag{2.18}$$

where

$$A^{\mathbf{B}} = A^A \mathbf{B}_A\,, \qquad A^{\mathbf{F}} = A^\alpha \mathbf{F}_\alpha. \tag{2.19}$$

The coset representatives are restricted to the form [4, 5]

$$\mathcal{G}(Z) = g(X)e^{\Theta}\,, \tag{2.20}$$

where Θ is an $\mathbf{F}$-valued object which may depend on X, i.e.

$$\Theta = \Theta^{\mathbf{F}} = \Theta^\alpha \mathbf{F}_\alpha = \theta^{\dot\alpha} e_{\dot\alpha}{}^\alpha(X)\mathbf{F}_\alpha\,. \tag{2.21}$$

To get a handle on the higher θ components of the superfields, consider the transformation

$$Z^M = \{X^\mu, \theta^{\dot\alpha}\} \to Z_t^M = \{X^\mu, t\theta^{\dot\alpha}\}\,, \qquad \text{so} \qquad \Theta \to t\Theta\,, \tag{2.22}$$

where t is a real parameter. Consider the Cartan forms as functions of the rescaled θ's

$$L_t = L(X, t\theta) = \mathcal{G}_t(Z_t)^{-1} d\mathcal{G}_t(Z_t)\,. \tag{2.23}$$

Differentiating the defining equation (2.2) with respect to t we obtain

$$\partial_t L_t = d\Theta + [L_t, \Theta]\,. \tag{2.24}$$

We split the Cartan forms as follows

$$L(X,\theta) = L_0(X) + \tilde{L}(X,\theta) \tag{2.25}$$

and make the coset representative has been split such that

$$L_0 = L_0^{\mathbf{B}} = g^{-1}(X)dg(X) \qquad \text{or} \qquad L_0^{\mathbf{F}} = 0\,. \tag{2.26}$$

Therefore the equation (2.24) reduces to

$$\partial_t \tilde{L}_t = D\Theta + [\tilde{L}_t, \Theta]\,, \tag{2.27}$$

where

$$D\Theta \equiv d\Theta + [L_0, \Theta] = d\Theta + [L_0^{\mathbf{B}}, \Theta]\,. \tag{2.28}$$

Taking a second derivative of (2.24) we obtain

$$\partial_t^2 \tilde{L}_t = \left[D\Theta + \left[\tilde{L}_t, \Theta\right], \Theta\right]\,. \tag{2.29}$$

Using (2.16) and some rearrangements of commutators we have

$$\partial_t^2 \tilde{L}_t^{\mathbf{F}} = \left[\Theta, \left[\Theta, \tilde{L}_t^{\mathbf{F}}\right]\right] \equiv \mathcal{M}^2 \tilde{L}_t^{\mathbf{F}}\,. \tag{2.30}$$

The matrix $\mathcal{M}^2$ acts on $\mathbf{G}$-valued objects A as

$$\mathcal{M}^2 A = [\Theta, [\Theta, A]]\,. \tag{2.31}$$

We can solve this oscillator type of equation by giving the initial conditions

$$\tilde{L}^{\mathbf{F}}_{(t=0)} = 0\,, \qquad \left(\partial_t \tilde{L}^{\mathbf{F}}\right)_{(t=0)} = D\Theta\,. \tag{2.32}$$

The solution reads

$$L_t^{\mathbf{F}} = \frac{\sinh t\mathcal{M}}{\mathcal{M}} D\Theta\,. \tag{2.33}$$

Using this result and the initial condition

$$\tilde{L}^{\mathbf{B}}_{(t=0)} = 0\,, \tag{2.34}$$

we solve the $\mathbf{B}$ component of the first order equation (2.27) to

$$L_t^{\mathbf{B}} = L_0^{\mathbf{B}} + 2\left[\frac{\sinh^2 t\mathcal{M}/2}{\mathcal{M}^2} D\Theta, \Theta\right]\,. \tag{2.35}$$

The Cartan forms are then obtained by setting $t = 1$. Thus we have reproduced the solution for the Cartan 1-forms of the superspace [5] associated with some superalgebra using the compact notation of the Lie-algebra valued objects and their commutators.

We use the same procedure to derive the higher order θ components of the Killing superfield $\Sigma(Z)$. Rescaling the coordinates Z again as in (2.22) we obtain

$$\partial_t \Sigma_t = [\Sigma_t, \Theta]\,, \tag{2.36}$$

by differentiating (2.15) w.r.t. t. We can solve this equations to all orders in θ in closed form in terms of the initial conditions

$$\begin{aligned} \Sigma_0^{\mathbf{F}}(X) = g^{-1}(X)\Upsilon_0^{\mathbf{F}} g(X)\,, \qquad \Sigma_0^{\mathbf{B}}(X) = g^{-1}(X)\Upsilon_0^{\mathbf{B}} g(X)\,, \\ (\partial_t \Sigma^{\mathbf{F}})_{(t=0)} = \left[\Sigma_0^{\mathbf{B}}(X), \Theta\right] \equiv \mathcal{B}\Theta\,. \end{aligned} \tag{2.37}$$

Taking a second derivative of (2.36) we get

$$\partial_t^2 \Sigma_t = [[\Sigma_t, \Theta], \Theta]\,, \tag{2.38}$$

For the fermionic direction using (2.16) this gives again an oscillator type equation. After rearranging the commutators we get

$$\partial_t^2 \Sigma_t^{\mathbf{F}} = \mathcal{M}^2 \Sigma_t^{\mathbf{F}}\,, \tag{2.39}$$

with $\mathcal{M}$ given in (2.31). The solution to this equation reads, with initial conditions (2.37)

$$\Sigma_t^{\mathbf{F}} = \cosh t\mathcal{M} \Sigma_0^{\mathbf{F}} + \left(\frac{\sinh t\mathcal{M}}{\mathcal{M}}\right) \mathcal{B}\Theta\,. \tag{2.40}$$

Taking this solution into the $\mathbf{B}$ component of (2.36) leads to

$$\Sigma_t^{\mathbf{B}} = \Sigma_0^{\mathbf{B}} + \left[\frac{\sinh t\mathcal{M}}{\mathcal{M}} \Sigma^{\mathbf{F}}, \Theta\right] + 2\left[\frac{\sinh^2 t\mathcal{M}/2}{\mathcal{M}^2} [L_0, \Theta], \Theta\right]\,. \tag{2.41}$$

The G-covariant Killing superfields are recovered at $t = 1$ in terms of the known θ-independent Killing fields $\Sigma_0^A(X)$.

To conclude let us summarize what we have now. Given any supercoset space G/H and the coset representative (i.e. choice of coordinates) of the bosonic subspace $g(X)$, we can construct the complete geometric superfields $L(Z)$ and Killing superfields $\Sigma(Z)$. By taking the expression (2.20) for the supercoset representative we have assumed that fermionic Cartan 1-forms vanish at vanishing θ, i.e. $L_0^{\mathbf{F}} = 0$. The usual Killing vectors, Killing spinors and compensating stability group transformations can be recovered by identifying the coset generators and stability group generators and use the transformation (2.13). We summerize here the relevant expressions for the Cartan 1-forms and G-covariant Killing fields in index notation, i.e. in terms of the structure constants of the $\mathbf{G}$ algebra,

$$\begin{aligned}
L^\alpha &= \left(\frac{\sinh \mathcal{M}}{\mathcal{M}} D\Theta\right)^\alpha, \qquad L^A = L_0^A + 2\Theta^\alpha f_{\alpha\beta}^A \left(\frac{\sinh^2 \mathcal{M}/2}{\mathcal{M}^2} D\Theta\right)^\beta, \\
\Sigma^\alpha &= (\cosh t\mathcal{M}\ \Sigma_0)^\alpha + \left(\frac{\sinh \mathcal{M}}{\mathcal{M}} \mathcal{B}\Theta\right)^\alpha, \\
\Sigma^A &= \Sigma_0^A + \Theta^\alpha f_{\alpha\beta}^A \left(\frac{\sinh \mathcal{M}}{\mathcal{M}}\ \Sigma_0 + 2\frac{\sinh^2 \mathcal{M}/2}{\mathcal{M}^2} \mathcal{B}\Theta\right)^\beta, \\
(\mathcal{B}\Theta)^\alpha &= \Theta^\beta \Sigma_0^A f_{A\beta}^\alpha\,, \\
(\mathcal{M}^2)^\alpha{}_\beta &= f_{A\gamma}^\alpha \Theta^\gamma \Theta^\delta f_{\delta\beta}^A\,.
\end{aligned} \tag{2.42}$$

The most difficult point in establishing a nice superspace and its symmetries is the choice of the coset representative for the bosonic subspace, i.e. choosing appropriate coordinates. In particular applications one could prefer having an explicit expression for the more 'familiar' H-covariant super Killing vector and spinor and the comensating H-transformation. We will derive these expressions in the next section in a special gauge.

3. H-covariant superisometries in Killing-spinor gauge

In this section we will recover the superisometries in H-covariant form which gives the transformation of the coordinates

$$\delta Z^M = -\Xi^M \tag{3.1}$$

and the corresponding compensating stability group transformation given by the parameter Λ^i in (2.12).

We consider the coset decomposition of the algebra

$$\mathbf{G} = \mathbf{K} \oplus \mathbf{H}\,, \tag{3.2}$$

where $\mathbf{K}$ collects the "coset generators" $\mathbf{K}_{\bar{M}}$ and $\mathbf{H}$ collects the stability algebra generators $\mathbf{H}_i$. The algebra decomposes into

$$\begin{aligned}
[\mathbf{K}, \mathbf{K}\} &= \mathbf{K} \oplus \mathbf{H}\,,\\
[\mathbf{H}, \mathbf{K}\} &= \mathbf{K}\,,\\
[\mathbf{H}, \mathbf{H}\} &= \mathbf{H}\,.
\end{aligned} \tag{3.3}$$

In the flat and near-horizon superspaces we deal with a *homogeneous* superspace which is not symmetric[4]. A symmetric space would have $[\mathbf{K}, \mathbf{K}\} = \mathbf{H}$. And we will consider only a *reductive* decomposition, $[\mathbf{H}, \mathbf{K}\} = \mathbf{K}$, which is satisfied by the flat and near-horizon superspaces. A more general form of the coset would also contain $[\mathbf{H}, \mathbf{K}\} = \mathbf{K} \oplus \mathbf{H}$. We split the Cartan 1-forms L and G-covariant parameters Σ according to (3.2)

$$\begin{aligned}
L &= E + \Omega = E^{\bar{M}}\mathbf{K}_{\bar{M}} + \Omega^i \mathbf{K}_i\,,\\
\Sigma &= \hat{\Xi} + \hat{\Lambda} = \hat{\Xi}^{\bar{M}}\mathbf{K}_{\bar{M}} + \hat{\Lambda}^i \mathbf{H}_i\,.
\end{aligned} \tag{3.4}$$

Since in this paper we are interested in maximally supersymmetric superspaces we will restrict ourselves to the case where

$$\mathbf{F} \subset \mathbf{K} \qquad \text{or} \qquad \mathbf{F} \cap \mathbf{H} = 0\,. \tag{3.5}$$

The G-covariant parameters $\{\hat{\Xi}^{\bar{M}}, \hat{\Lambda}^i\}$ are related to the H-covariant ones through eqn. (2.13),

$$\hat{\Xi}^{\bar{M}} = \Xi^M E_M{}^{\bar{M}}\,, \qquad \hat{\Lambda}^i = \Lambda^i + \Xi^M \Omega_M{}^i\,. \tag{3.6}$$

[4]We thank P. Howe for pointing this out.

We can split the bosonic generators of the previous section into

$$\mathbf{B} = \{\mathbf{P}_a, \mathbf{M}_i\}, \qquad P_a \in \mathbf{K} \text{ and } M_i \in \mathbf{H}. \tag{3.7}$$

Before we write down the full θ-dependent parameters, we repeat the $\theta = 0$ killing equations. We denote the Cartan 1-forms L_0^Λ by

$$\begin{aligned}
\psi \equiv L_0^\alpha \mathbf{F}_\alpha &= \psi^\alpha \mathbf{F}_\alpha = dx^\mu \psi_\mu^\alpha \mathbf{F}_\alpha\,,\\
e \equiv L_0^a \mathbf{P}_a &= e^a \mathbf{P}_a = dx^\mu e_\mu{}^a \mathbf{P}_a\,,\\
\omega \equiv L_0^i \mathbf{H}_i &= \omega^i \mathbf{H}_i = dx^\mu \omega_\mu^i \mathbf{H}_i\,,
\end{aligned} \tag{3.8}$$

where in the construction outlined above ψ has been chosen to vanish. The names have not been chosen at will since they are the vielbein e^a, the gravitino ψ^a and the spin connection ω^i of the geometry, which is a solution to a supergravity theory. We will clarify the relation between supergravity solutions and supercoset spaces in later sections. We introduce also the more familiar parameters

$$\begin{aligned}
\epsilon &\equiv \hat{\Xi}_0^\alpha \mathbf{F}_\alpha = \epsilon^\alpha \mathbf{F}_\alpha\,,\\
\hat{\xi} &\equiv \hat{\Xi}_0^a \mathbf{P}_a = \xi^a \mathbf{P}_a = \xi^\mu e_\mu{}^a \mathbf{P}_a\,,\\
\hat{\ell} &\equiv \hat{\Lambda}_0^i \mathbf{M}_i = \hat{\ell}^i \mathbf{M}_i = (\ell^i + \xi^\mu \omega_\mu{}^i)\mathbf{M}_i\,.
\end{aligned} \tag{3.9}$$

The θ-independent part of (2.14) splits up into ($\psi = 0$)

$$\begin{aligned}
0 &= (d\epsilon + [\omega, \epsilon] + [e, \epsilon])^\alpha \mathbf{F}_\alpha\,,\\
0 &= (d\xi + [\omega, \hat{\xi}] + [e, \hat{\ell}] + [e, \hat{\xi}])^a \mathbf{P}_a\,,\\
0 &= (d\hat{\ell} + [\omega, \hat{\ell}] + [\omega, \hat{\xi}] + [e, \hat{\xi}])^i \mathbf{M}_i\,.
\end{aligned} \tag{3.10}$$

We rewrite the last two equations using (3.9) and obtain the set of killing-equations

$$0 = \delta\psi^\alpha \mathbf{F}_\alpha = (d\epsilon + [\omega, \epsilon] + [e, \epsilon])^\alpha \mathbf{F}_\alpha\,, \tag{3.11}$$

$$0 = \delta e^a \mathbf{P}_a = (\mathcal{L}_\xi e^a + [e, \ell]^a)\mathbf{P}_a\,, \tag{3.12}$$

$$0 = \delta\omega^i \mathbf{M}_i = (\mathcal{L}_\xi \omega^i + d\ell^i + [\omega, \ell]^i)\mathbf{M}_i\,. \tag{3.13}$$

These equations give the non-vanishing values of Killing spinors $\epsilon^\alpha(X)$, Killing vectors $\xi^\nu(X)$ and compensating Lorentz transformation $\ell^i(X)$ at order θ^0. From the supergravity point of view the meaning of these equations is that this set of transformations leaves the supergravity solution invariant, i.e. under these supersymmetry transformations, general coordinate transformations and Lorentz transformations, the gravitino, vielbeins and spin connections, do not change.

Before we go on to derive Ξ^M and Λ^i to all orders in θ, we will take a special value of $e_{\dot\alpha}{}^\alpha$ related to the solution of the killing spinor equation (3.11). It is known that the solution to (3.11) can be written as

$$\epsilon^\alpha(X) = \epsilon_0^{\dot\alpha} \mathcal{K}(X)_{\dot\alpha}{}^\alpha\,, \tag{3.14}$$

where ϵ_0 is a constant spinor. As the matrix $e_{\dot\alpha}{}^{\alpha}$ was left unspecified until now we can still *choose* it to be equal to $\mathcal{K}$, called the Killing-spinor gauge. Doing so we find

$$e_{\dot\alpha}{}^{\alpha} = \mathcal{K}_{\dot\alpha}{}^{\alpha} \quad \Longrightarrow \quad (D\Theta) = (d\theta)^{\dot\alpha}\mathcal{K}_{\dot\alpha}{}^{\alpha}\mathbf{F}_{\alpha}\,. \tag{3.15}$$

This leads to simplifications of the Cartan 1-forms

$$L_{\mu}{}^{\Lambda}(X,\theta) = L_{0\mu}{}^{\Lambda}(X)\,. \tag{3.16}$$

In this gauge we obtain

$$\begin{aligned}
\Xi^{\alpha}\mathbf{F}_{\alpha} &\equiv \Xi^{\dot\alpha}\mathcal{K}_{\dot\alpha}{}^{\alpha}\mathbf{F}_{\alpha} = (\mathcal{M}\coth\mathcal{M}\epsilon) + (\mathcal{B}\Theta)\,,\\
\Xi^{a}\mathbf{P}_{a} &\equiv \Xi^{\mu}e_{\mu}{}^{a}\mathbf{P}_{a} = \xi^{a}\mathbf{P}_{a} + \left[\frac{\tanh\mathcal{M}/2}{\mathcal{M}}\epsilon,\Theta\right]^{a}\mathbf{P}_{a}\,,\\
\Lambda^{i} &= \ell^{i}\mathbf{M}_{i} + \left[\frac{\tanh\mathcal{M}/2}{\mathcal{M}}\epsilon,\Theta\right]^{a} e_{a}{}^{\mu}\omega_{\mu}{}^{i}\mathbf{M}_{i} + \left[\frac{\tanh\mathcal{M}/2}{\mathcal{M}}\epsilon,\Theta\right]^{i}\mathbf{M}_{i}\,.
\end{aligned} \tag{3.17}$$

This concludes the derivation of the superisometries. On the coordinates they act as

$$\delta X^{\mu} = -\Xi^{\mu}(X,\theta)\,, \qquad \delta\theta^{\dot\alpha} = -\Xi^{\dot\alpha}(X,\theta)\,. \tag{3.18}$$

4. Superisometries of maximally supersymmetrc vacua

In this section we will present the superisometries for the two maximally supersymmetric vacua, i.e. flat space, where G is the super Poincaré group and $AdS_{p+2}\times S^{d-p-2}$ with G the relevant extended superconformal group.

4.1 Flat superspace

As a check on the general results obtained above we apply our formulae to the flat superspace. The coset starts with the super Poincaré group G. The algebra is given by

$$\begin{aligned}
[M_{ab},M_{cd}] &= \eta_{a[c}M_{d]b} - \eta_{b[c}M_{d]a}\,,\\
[P_a,M_{bc}] &= \eta_{a[b}P_{c]}\,,\\
[M_{ab},Q_{\alpha}] &= -\frac{1}{4}(\gamma_{ab}Q)_{\alpha}\,,\\
\{Q_{\alpha},Q_{\beta}\} &= (\gamma^{a})_{\alpha\beta}P_a\,.
\end{aligned} \tag{4.1}$$

We make the following split

$$\mathbf{H} = \{M_{ab}\}\,, \qquad \text{and} \qquad \mathbf{K} = \{P_a, Q_{\alpha}\}\,. \tag{4.2}$$

The indices of the previous sections are thus

$$\Lambda = \{a,(ab),\alpha\}\,, \qquad A = \{a,(ab)\}\,, \qquad i = \{(ab)\}\,, \qquad \bar{M} = \{a,\alpha\}\,. \tag{4.3}$$

The space-time fields are given by

$$e_\mu^a = \delta_\mu^a\,, \qquad \psi_\mu = 0\,, \qquad \omega_\mu^{ab} = 0\,, \tag{4.4}$$

and the solutions to the space-time killing equations is

$$\xi^\mu = a^\mu + \lambda_M^{\mu\nu} x_\nu\,, \qquad \epsilon^\alpha(x) = \epsilon_0^\alpha\,, \qquad \ell^{ab} = \lambda_M^{\mu\nu}\delta_\mu^a \delta_\nu^b\,, \tag{4.5}$$

where a^μ, $\lambda_M^{\mu\nu}$ and ϵ_0^α are constant parameters. The matrix $\mathcal{K}_{\dot\alpha}{}^\beta = \delta_{\dot\alpha}^\beta$ and the matrix $\mathcal{M}$ vanishes. The Cartan 1-forms are given by

$$E^\alpha = d\theta^\alpha\,, \qquad E^a = dx^a + \bar\theta\gamma^a d\theta\,. \tag{4.6}$$

We read off that

$$\begin{aligned} \Xi^\mu &= a^\mu + \lambda_M^{\mu\nu} x_\nu + \bar\theta\gamma^\mu\epsilon_0\,, \\ \Xi^\alpha &= \epsilon_0^\alpha + \frac{1}{4}(\lambda_M\cdot\gamma\theta)^\alpha\,, \end{aligned} \tag{4.7}$$

which gives the well-known superspace and superisometries.

4.2 $AdS_{p+2}\times S^{d-p-2}$ superspace

To construct this superspace we start from the superconformal group G which has $\mathrm{SO}(p+1,2)\times\mathrm{SO}(d-p-1)$ as its bosonic subgroup. For this supercoset the stability group H is the product group $\mathrm{SO}(p+1,1)\times\mathrm{SO}(d-p-2)$, which is purely bosonic. In particular we will derive explicit expressions for the Killing vectors ξ^μ and Killing spinors ϵ^α as well as the compensating H-transformation parameters ℓ^{ab} for the prototypical brane solutions to various supergravity theories which have a near-horizon geometry of the form $AdS_{p+2}\times S^{d-p-2}$ and are summarized in table 1. The last two solutions in this table can be obtained as intersecting brane solutions.

The bosonic isometries, which form the group $\mathrm{SO}(p+1,2)\times\mathrm{SO}(d-p-1)$, can be treated in a uniform way for all cases in table 1. The geometry is described by the product space metric [13]

$$ds^2 = dx^\mu g_{\mu\nu} dx^\nu = g^{AdS} + g^S\,, \tag{4.8}$$

supergravity	brane solution	supergroup G
$d=11$ sugra	$M2$ brane ($p=2$)	$\mathrm{OSp}(8\|4)$
$d=11$ sugra	$M5$ brane ($p=5$)	$\mathrm{OSp}(6,2\|4)$
$d=10$ IIB sugra	$D3$ brane ($p=3$)	$\mathrm{SU}(2,2\|4)$
$d=6$ $(2,0)$ sugra	self-dual string ($p=1$)	$\mathrm{SU}(1,1\|2)^2$
$d=4$ $N=2$ sugra	Reissner-Nordstrøm black hole ($p=0$)	$\mathrm{SU}(1,1\|2)$

Table 1: Supergravity brane solutions with $AdS_{p+2}\times S^{d-p-2}$.

where

$$
\begin{aligned}
g^{AdS} &= \left(\frac{r}{R}\right)^{2/w} dx^m \eta_{mn} dx^n + \left(\frac{R}{r}\right)^2 dr^2\,,\\
g^S &= R^2 d\Omega^2\,.
\end{aligned}
\tag{4.9}
$$

Besides the metric there are also the non-trivial forms which will be introduced in due course and are proportional to the volume forms of the AdS_{p+2} and S^{d-p-2} spaces. The coordinates X^μ are split into AdS coordinates $X^{\hat{m}} = \{x^m, r\}$, $m = 0,\ldots,p$ and coordinates on the sphere, here we take angular coordinates $X^{m'} = \{\phi^{m'}\}$, $m' = 1,\ldots,d-p-2$. $d\Omega^2$ is the metric on the unit sphere. The number $w = \frac{p+1}{d-p-3}$ has a fixed value that gives the ratio between the radius of the AdS space and the sphere S for the specific supergravity solutions.

The Killing vectorfield $\xi \equiv \xi^\mu \partial_\mu$ is the solution to

$$
\mathcal{L}_\xi g = 0\,. \tag{4.10}
$$

Due to the fact that we are dealing with a product geometry, in the coordinates given above this equation splits into

$$
\mathcal{L}_{\xi_{AdS}} g^{AdS} = 0\,, \qquad \mathcal{L}_{\xi_S} g^S = 0\,. \tag{4.11}
$$

The components of the killing-vector field ξ_{AdS} of the AdS_{p+2} space are given by [13]

$$
\begin{aligned}
\xi^m_{AdS} &= \xi^m_{conf}(x) + (wR)^2 \left(\frac{R}{r}\right)^{2/w} \Lambda^m_K\\
&= a^m + \lambda^{mn}_M x_n + \lambda_D x^n + (x^2\Lambda^m_K - 2x^m x\cdot\Lambda_K) + (wR)^2\left(\frac{R}{r}\right)^{2/w}\Lambda^m_K\,,\\
\xi^r_{AdS} &= -w\Lambda_D(x) r\,,
\end{aligned}
\tag{4.12}
$$

where

$$
\Lambda_D(x) = \frac{1}{p+1}\partial_m \xi^m_{conf} = \lambda_D - 2x\cdot\Lambda_K\,. \tag{4.13}
$$

The compensating transformation of the stability subgroup $\mathrm{SO}(p+1,1)$ is

$$
\begin{aligned}
\ell^{\bar{m}\bar{n}} &= \Lambda^{mn}_M(x)\,\delta^{\bar{m}}_m \delta^{\bar{n}}_n\,,\\
\ell^{\bar{m}\bar{r}} &= -2(wR)\left(\frac{R}{r}\right)^{1/w} \Lambda^m_K\,\delta^{\bar{m}}_m\,,
\end{aligned}
\tag{4.14}
$$

where

$$
\Lambda^{mn}_M(x) = -2\partial^{[m}\xi^{n]}_{conf} = \lambda^{mn}_M - 4x^{[m}\Lambda^{n]}_K\,. \tag{4.15}
$$

This is obtained as a solution to

$$
0 = \mathcal{L}_{\xi_{AdS}} e^{\bar{\hat{m}}} - \ell^{\bar{\hat{m}}\bar{\hat{n}}} e_{\bar{\hat{n}}}\,. \tag{4.16}
$$

In these Killing vectors and compensating transformations, we identify the conformal Killing vectors ξ^m_{conf} and conformal 'compensating' transformations with parameters $\Lambda_D(x)$ and $\Lambda^{mn}_M(x)$.

On S^{d-p-2} the killing vectors $\xi^{m'}_S$ as well as the compensating transformation $\ell^{\bar{m}'\bar{n}'}$ from the stability subgroup $\mathrm{SO}(d-p-2)$ are somewhat complicated functions of the angles. These transformations can be expressed in terms of the R-symmetry subgroup of the superconformal group which is the symmetry of the surface

$$X^{\hat{m}'}\delta_{\hat{m}'\hat{n}'}X^{\hat{m}'} = R^2 \tag{4.17}$$

where $\hat{m}'$ takes values $1,\ldots,d-p-1$. The $\mathrm{SO}(d-p-1)$ invariance is linearly realized on the $X^{\hat{m}'}$ coordinates,

$$\delta X^{\hat{m}'} = -\Lambda^{\hat{m}'}{}_{\hat{n}'}X^{\hat{n}'}\,; \qquad \Lambda_{\hat{m}'\hat{n}'} = -\Lambda_{\hat{n}'\hat{m}'}\,. \tag{4.18}$$

This surface condition can be solved in terms of hyperspherical coordinates $\phi^{m'}$

$$X^{\hat{m}'} = \left\{R\cos\phi^1\,,\ldots\,,R\cos\phi^{d-p-2}\prod_{i=1}^{d-p-3}\sin\phi^i\,,R\prod_{i=1}^{d-p-2}\sin\phi^i\right\}\,, \tag{4.19}$$

with

$$\phi^{m'} = \arctan\left[\frac{\left(\sum_{k'=m'+1}^{d-p-1}(X^{k'})^2\right)^{1/2}}{X^{m'}}\right]\,. \tag{4.20}$$

The change of angular variables $\delta\phi^{m'} = -\xi^{m'}_S$ which preserves the metric on the sphere can be deduced from (4.20). These transformations do not preserve the vielbein forms and one has to find also the compensating transformation which preserves the vielbeins. It is important that all R-symmetry transformations depend only on angles and global parameters $\Lambda^{\hat{m}'}{}_{\hat{n}'}$. As an example consider here the case of S^2 with two angles, $d\Omega^2 = (d\phi^1)^2 + (\sin\phi^1 d\phi^2)^2$, i.e. $m' = 1,2$ and $\hat{m}' = 1,2,3$. One finds that the SO(3) isometry (R-symmetry) is realized on the two angles of the sphere as

$$\begin{aligned}\xi^{\phi_1} &= -\cos\phi_2\Lambda^{12} - \sin\phi_2\Lambda^{13}\,,\\ \xi^{\phi_2} &= \cot\phi_1\sin\phi_2\Lambda^{12} - \cot\phi_1\cos\phi_2\Lambda^{13} - \Lambda^{23}\,,\end{aligned} \tag{4.21}$$

and the compensating SO(2) transformation is

$$\ell^{\bar{\phi}_1\bar{\phi}_2} = \sin^{-1}\phi_1(\sin\phi_2\Lambda^{12} - \cos\phi_2\Lambda^{13})\,. \tag{4.22}$$

It is not so difficult to see that $\frac{\partial}{\partial\phi^{d-p-2}}$ will always be a Killing vector and therefore there will always be conservation of momentum in this angular direction. This concludes the derivation of the bosonic isometries.

The Killing spinors cannot be treated in the same uniform way because properties of spinors depend on the dimension and signature. However, a detailed study of the relevant Killing spinor equations (vanishing transformation of fermions at vanishing fermions in the supergravity theory) for each case in table 1 reveals a general structure of the Killing spinors. The difference between all cases is in the split of the generic fermions by some specific projection operator. First we will give the supergravity Killing-spinor equations.

1. *The* $G = \mathrm{OSp}(M|N)$ *cases*

(a) *$M2$ and $M5$ brane solutions to $d = 11$ supergravity*
Besides the metric (4.9), the near-horizon $M2$ and $M5$ solutions have a non-vanishing 4-form field strength and its dual 7-form field strength, given by

$$\begin{aligned} &\mathrm{M2} \qquad F^4_{\bar{\bar{m}}_1 \ldots \bar{\bar{m}}_4} = -\frac{6}{R}\epsilon_{\bar{\bar{m}}_1 \ldots \bar{\bar{m}}_4} \qquad F^7_{\bar{m}'_1 \ldots \bar{m}'_7} = \frac{6}{R}\epsilon_{\bar{m}'_1 \ldots \bar{m}'_7}\,, \\ &\mathrm{M5} \qquad F^7_{\bar{\bar{m}}_1 \ldots \bar{\bar{m}}_7} = \frac{3}{R}\epsilon_{\bar{\bar{m}}_1 \ldots \bar{\bar{m}}_7}\,, \qquad F^4_{\bar{m}'_1 \ldots \bar{m}'_4} = \frac{3}{R}\epsilon_{\bar{m}'_1 \ldots \bar{m}'_4}\,. \end{aligned} \tag{4.23}$$

The Killing spinor equation is given by the vanishing variation of the gravitino at vanishing gravitino [16]

$$0 = \delta\psi_\mu = \partial_\mu \epsilon + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\epsilon + \frac{1}{288}e_\mu{}^a \left(\Gamma_a{}^{bcde} - 8\delta_a^{[b}\Gamma^{cde]}\right) F_{bcde}\epsilon\,. \tag{4.24}$$

The spinor ϵ is a 32 component Majorana spinor. To solve these equations it turns out that it is useful to introduce projections of the spinors

$$\epsilon_\pm = \mathcal{P}_\pm \epsilon = \frac{1}{2}(1 \pm \Gamma^{01\ldots p})\epsilon\,. \tag{4.25}$$

2. *The* $G = \mathrm{SU}(M|N)$ *cases*

(a) *D3 brane solution to $d = 10$ IIB supergravity*
The $D3$ brane solution to 10 dimensional IIB supergravity is given by the metric (4.9) with $p = 3$, $w = 1$, the self dual 5-form with components

$$F_{\bar{\bar{m}}_1 \ldots \bar{\bar{m}}_5} = \frac{4}{R}\epsilon_{\bar{\bar{m}}_1 \ldots \bar{\bar{m}}_5}\,, \qquad F_{\bar{m}'_1 \ldots \bar{m}'_5} = \frac{4}{R}\epsilon_{\bar{m}'_1 \ldots \bar{m}'_5}\,, \tag{4.26}$$

and a *constant* dilaton ϕ. The fermions and all other forms vanish.
With this solution the Killing spinor equations [17] can be cast in the form

$$\begin{aligned} &0 = \delta\lambda = 0\,, \\ &0 = \delta\psi = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}\epsilon - \frac{i}{16\cdot 5!}e_\mu{}^a F^+_{bcdef}\Gamma^{bcdef}\Gamma_a\epsilon\,, \end{aligned} \tag{4.27}$$

where the dilatino λ is a complex are left-handed Weyl spinors (or two Majorana-Weyl spinors), i.e. $\Gamma_{11}\lambda = \lambda$. The parameters ϵ^i and the gravitini ψ^i_μ are right-handed Weyl spinors, $\Gamma_{11}\epsilon = -\epsilon$ and $\Gamma_{11}\psi_\mu = -\psi_\mu$. The variation of the dilatino vanishes trivially for vanishing fermions since it transforms into the derivative of the dilaton and it does not transform into the self-dual 5-form. To solve the Killing spinor equation we introduce

$$\epsilon_{\pm} = \mathcal{P}_{\pm}\epsilon = \frac{1}{2}(\epsilon \pm i\Gamma^{0123}\epsilon)\,, \tag{4.28}$$

(b) *self-dual string solution to $d = 6$ $(0,2)$ supergravity*
The chiral $d = 6$ $(0,2)$ pure supergravity theory contains [18] the graviton $g_{\mu\nu}$, 4 chiral gravitini ψ^i_μ and 5 self-dual tensors $B^{a'}_{\mu\nu}$, where a' is an SO(5) vector index. This theory has also been known as pure $N = 4b$ supergravity. The self-dual string solution to this theory has the metric (4.9) with $p = 1$, $w = 1$, the self dual 3-form with components

$$F^{a'}_{\bar{\bar{m}}_1\bar{\bar{m}}_2\bar{\bar{m}}_3} = \frac{2}{R}\epsilon_{\bar{\bar{m}}_1\bar{\bar{m}}_2\bar{\bar{m}}_3}\delta^{a'}_{1'}\,, \qquad F^{a'}_{\bar{m}'_1\bar{m}'_2\bar{m}'_3} = \frac{2}{R}\epsilon_{\bar{m}'_1\bar{m}'_2\bar{m}'_3}\delta^{a'}_{1'} \tag{4.29}$$

and the gravitini vanish.
With this solution the vanishing variation of the gravitini are [18]

$$0 = \delta\psi^i_\mu = \partial_\mu\epsilon^i + \frac{1}{4}\omega^{ab}_\mu\Gamma_{ab}\epsilon^i - \frac{1}{8}e_\mu{}^a F^1_{abc}\Gamma^{bc}(\Gamma')^i{}_j\epsilon^j\,. \tag{4.30}$$

Here ϵ^i (and also ψ^i_μ) is a righthanded ($\Gamma_7\epsilon^i = -\epsilon^i$) symplectic Majorana-Weyl spinor. The USp(4) index i is raised and lowered by $\epsilon^i = \Omega^{ij}\epsilon_j$ and $\epsilon_i = \epsilon^j\Omega_{ji}$, where Ω_{ij} is the antisymmetric USp(4) metric and we define $\Omega^{ik}\Omega_{jk} = \delta^i{}_j$. The matrix $\Gamma' \equiv \Gamma'_{1'}$ is an SO(5) Γ-matrix satisfying $(\Gamma')^2 = 1$ and $[\Gamma^a, \Gamma'] = 0$. For the self-dual string solution we introduce the projections

$$\epsilon_{\pm} \equiv \epsilon^i_{\pm} = \mathcal{P}_{\pm}\epsilon^i = \frac{1}{2}\left(\delta^i{}_j \pm \Gamma^{01}(\Gamma')^i{}_j\right)\epsilon^j\,. \tag{4.31}$$

(c) *Reissner-Nordstrøm black hole solution to $d = 4$ $N = 2$ supergravity*
The metric (4.9) for $p = 0$, $w = 1$ and $d = 4$ is the near-horizon geometry of an extremal Reissner-Nordstrøm black hole solution in isotropic coordinates, where R is proportional to the mass of the black hole . This solution can be obtained as a 1/2 BPS solution to $d = 4$ $N = 2$ pure supergravity. We will restrict to the case where the black hole is electrically charged and therefore we restrict to the case that the graviphoton field strength has components

$$F_{\bar{\bar{m}}_1\bar{\bar{m}}_2} = -\frac{1}{R}\epsilon_{\bar{\bar{m}}_1\bar{\bar{m}}_2}\,. \tag{4.32}$$

The Killing-spinor equation is

$$0 = \delta\psi^i_\mu = \partial_\mu\epsilon^i + \frac{1}{4}\omega^{ab}_\mu\Gamma_{ab}\epsilon^i + e_\mu{}^a F^-_{ab}\Gamma^b\varepsilon^{ij}\epsilon_j\,, \tag{4.33}$$

where ϵ^i (and also ψ^i_μ) is a generic doublet ($i = 1, 2$) of Majorana spinors satisfying $-i(\epsilon_i)^\dagger\Gamma_0 = (\epsilon^i)^T\mathcal{C}$ with $\mathcal{C}$ the 4 dimensional charge conjugation matrix. The place of the index i denotes the chirality of the spinor, e.g. $\Gamma_5\epsilon^i = \epsilon^i$ and $\Gamma_5\epsilon_i = -\epsilon_i$. F^-_{ab} is the anti-self dual part of F_{ab}. To solve the Killing-spinor equation for this case we introduce

$$\epsilon_\pm \equiv \epsilon^i_\pm = \mathcal{P}_\pm\epsilon^i = \frac{1}{2}(\epsilon^i \pm \Gamma^0\varepsilon^{ij}\epsilon_j)\,. \tag{4.34}$$

Using the appropriate projector for each case at hand we can rewrite the killing spinor equations as

$$\begin{aligned}
0 &= \partial_m\epsilon_+ + \frac{1}{wR}\left(\frac{r}{R}\right)^{1/w}\Gamma_m\Gamma_r\epsilon_-\,, &(4.35)\\
0 &= \partial_m\epsilon_-\,, &(4.36)\\
0 &= \partial_r\epsilon_\pm \mp \frac{1}{2w}\frac{1}{r}\epsilon_\pm\,, &(4.37)\\
0 &= \nabla_{m'}\epsilon_\pm \mp \frac{1}{2R}\Gamma_r\tilde{e}_{m'}{}^{\bar{m}'}\Gamma_{m'}\epsilon_\pm\,, &(4.38)
\end{aligned}$$

where $\nabla_{m'}$ is the covariant derivative and $\tilde{e}_{m'}{}^{\bar{m}'}$ is the vielbein of S^{d-p-2}. The explicit solutions to the Killing-spinor equations (4.38) on the sphere have been derived in [14]. The full solution to the Killing-spinor equations is ($\epsilon = \epsilon_+ + \epsilon_-$)

$$\begin{aligned}
\epsilon_- &= -(wR)\left(\frac{R}{r}\right)^{1/2w}\Gamma_r u(\phi^{m'})\eta\,,\\
\epsilon_+ &= \left(\frac{r}{R}\right)^{1/2w} u(\phi^{m'})\,(\epsilon_0 + x\cdot\Gamma\eta)\,,
\end{aligned} \tag{4.39}$$

where ϵ_0 and η are constant spinors, satisfying

$$\epsilon_0 = \mathcal{P}_+\epsilon_0\,, \qquad \eta = \Gamma_r\mathcal{P}_-\Gamma_r\eta\,. \tag{4.40}$$

The function $u(\phi^{m'})$ results from the Killing-spinor equation on the sphere and are given by [14]

$$u(\phi^{m'}) = \left(\cos\frac{\phi^1}{2} + \Gamma_{r1'}\sin\frac{\phi^1}{2}\right)\prod_{k'=2}^{d-p-2}\left(\cos\frac{\phi^{k'}}{2} + \Gamma_{(k'-1)k'}\sin\frac{\phi^{k'}}{2}\right)\,. \tag{4.41}$$

Interestingly enough again the superconformal Killing spinor in Minkowski flat background

$$\epsilon(x) = (\epsilon + x\cdot\Gamma\eta)\,, \tag{4.42}$$

where ϵ is the parameter of rigid supersymmetry and η is the parameter of rigid conformal supersymmetry, appears in this form of the Killing spinors. This Killing spinor was already introduced in the pioneering papers on supersymmetry by Wess and Zumino [19]. To conclude this derivation we note that we did not have to choose a particular split of the Γ-matrices in terms of tensor products of lower dimensional Γ-matrices. Also the Killing spinors are given by projections of the generic spinor of the background supergravity theory.

Now we are ready to present the superisometries of $AdS_{p+2} \times S^{d-p-2}$ superspace in the Killing spinor gauge

$$-\delta_{AdS}\theta^{\dot\alpha} = (\mathcal{K}^{-1})^{\dot\alpha}{}_{\alpha} \left[(\mathcal{M} \coth \mathcal{M}\epsilon)^{\alpha} + (\theta\mathcal{K})^{\beta}\Sigma_0^A f_{A\beta}^{\alpha} \right] , \tag{4.43}$$

$$-\delta_{AdS}x^{\mu} = \xi^{\mu}(x) + \left[\frac{\tanh \mathcal{M}/2}{\mathcal{M}}\epsilon \right]^{\beta} (\theta\mathcal{K})^{\alpha} f_{\alpha\beta}^{a} e_a{}^{\mu}(x) \tag{4.44}$$

and the compensating H transformation is

$$\Lambda^{ab} = \ell^{ab} + \left(\frac{\tanh \mathcal{M}/2}{\mathcal{M}}\epsilon \right)^{\beta} (\theta\mathcal{K})^{\alpha} \left(f_{\alpha\beta}^{(ab)} + e_c^{\mu}\omega_{\mu}^{ab} f_{\alpha\beta}^{c} \right) . \tag{4.45}$$

Here ϵ is the Killing spinor from (4.39), $\mathcal{K}$ is defined in (3.14) and can be read off from (4.39), ξ^{μ} is given in (4.12) and (4.21), ℓ^{ab} in (4.14) and (4.22) and Σ_0^A is given by (2.13) and

$$(\mathcal{M}^2)^{\alpha}{}_{\beta} = f_{A\gamma}^{\alpha} (\theta\mathcal{K})^{\gamma} (\theta\mathcal{K})^{\delta} f_{\delta\beta}^{A} . \tag{4.46}$$

Here the formulae have been given in d dimensional language. To get the explicit form of the structure constants one can then split the $(d-1,1)$ spinor indices into $(p+1,1)$ and $d-p-2$ spinor indices. These are then the indices of the spinor representation of the two bosonic subalgebras listed in table 1. Alternatively, as was done in [6] one can rewrite the algebra in d-dimensional language. This basis of the algebra follows naturally from the supergravity solution and this is the subject of the next section.

5. Superisometries from supergravity decomposition of the algebra

Supergravity superspace describes the maximally supersymmetric vacua in a way which allows to use the cartesian coordinates in which the metric of the $AdS_{p+2} \times S^{d-p-2}$ space is presented in a form where it is not a product space

$$ds^2 = \left(\frac{r}{R} \right)^{2/w} dx^m \eta_{mn} dx^n + \left(\frac{R}{r} \right)^2 dy^I \delta_{IJ} dy^J . \tag{5.1}$$

This form of the metric is obtained from (4.9) by combining $\{r, \phi^{m'}\}$ into cartesian y^I. The advantage of these coordinates is that the R-symmetry SO$(d-p-1)$ acts linearly on them [13]. The Killing vectors are

$$\begin{aligned}
\xi^m &= a^m + \lambda_M^{mn} x_n + \lambda_D x^n + (x^2 \Lambda_K^m - 2x^m x \cdot \Lambda_K) + (wR)^2 \left(\frac{R}{r}\right)^{2/w} \Lambda_K^m \,, \\
\xi^I &= -w\Lambda_D(x) y^I + \Lambda_R^{IJ} y_J \,,
\end{aligned} \tag{5.2}$$

To obtain the compensating H-transformation, we compute the solution to

$$0 = \mathcal{L}_{\hat{\xi}} e^a - \Lambda^{ab} e_b \,. \tag{5.3}$$

where e^a are the vielbein 1-forms derived from (4.9) and the indices a split into $\{\bar{m}, \bar{I}\}$. The parameter of the compensating H-transformation in (x, y) coordinates is given by

$$\begin{aligned}
\ell^{\bar{m}\bar{n}} &= \Lambda_M^{mn}(x)\, \delta_m^{\bar{m}} \delta_n^{\bar{n}} \,, \\
\ell^{\bar{m}\bar{I}} &= -2\frac{(wR)}{r} \left(\frac{R}{r}\right)^{1/w} \Lambda_K^m y^I \, \delta_m^{\bar{m}} \delta_I^{\bar{I}} \,, \\
\ell^{\bar{I}\bar{J}} &= \Lambda_R^{IJ} \, \delta_I^{\bar{I}} \delta_J^{\bar{J}} \,.
\end{aligned} \tag{5.4}$$

The Killing-spinors are

$$\begin{aligned}
\epsilon_- &= (wR) \left(\frac{R}{r}\right)^{1/2w} \frac{y \cdot \Gamma}{r} \eta \,, \\
\epsilon_+ &= \left(\frac{r}{R}\right)^{1/2w} (\epsilon_0 - x \cdot \Gamma \eta) \,.
\end{aligned} \tag{5.5}$$

These equations above give us nice and simple initial conditions but we have also to solve for the geometry and for the full superisometries. To do that we will start with the supergravity superspace.

In cartesian coordinates the on shell superfields, defining the torsions and curvature are not constant anymore, as in the product space case, where the form fields are just given by the volume forms of AdS space or a sphere, but are only covariantly constant [7]. We will therefore start from supergravity superspace and we will see that apart from the fact that we do not have structure constants as in supercoset case, the rest works the same way as before with structure "constants", which are only covariantly constant. For the 11 dimensional cases the OSp$(8|4)$ and OSp$(6,2|4)$ have been rewritten in terms of the solution to the supergravity formfields F in [6], which yields the algebra in 11-dimensional language. Then the supercoset method was used to derived the geometric superfields. In [8] it was shown that this was completely equivalent with supergravity superspace. Here we will develop the exact relation between the coset superspace and supergravity superspace.

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In what follows we show how to write the algebras in terms of supergravity torsion and curvature components. The torsion and curvature constraints of the supergravity superspace

$$
\begin{aligned}
0 &= \left(dE^\alpha - \frac{1}{4}(\Omega\cdot\gamma E)^\alpha - E^\beta E^a \mathcal{T}^\alpha_{a\beta}\right) Q_\alpha\,,\\
0 &= \left(dE^a - E^b \Omega_b{}^a - \frac{1}{2}E^\beta E^\alpha \mathcal{T}^a_{\alpha\beta}\right) P_a\,,\\
0 &= \left(d\Omega^{ab} - \Omega^a{}_c \Omega^{cb} - \frac{1}{2}E^d E^c \mathcal{R}_{cd}{}^{ab} - \frac{1}{2}E^\beta E^\alpha \mathcal{R}_{\alpha\beta}{}^{ab}\right) M_{ab}\,,
\end{aligned}
\tag{5.6}
$$

take the form of Maurer-Cartan equations (2.3). We have multiplied the constraints with generators $\{Q_\alpha, P_a, M_{ab}\}$. In general this is not the case, only when the curvatures and torsions are covariantly constant. To show this we will first translate to the notation used in supercoset case

$$
\begin{aligned}
L^A &= \{E^a, \Omega^{ab}\}\,, \qquad \text{so} \qquad A = \{a, (ab)\}\\
L^\alpha &= \{E^\alpha\}\,.
\end{aligned}
\tag{5.7}
$$

which come with the generators

$$
B_A = \{P_a, M_{ab}\}\,, \qquad F_\alpha = \{Q_\alpha\}\,. \tag{5.8}
$$

Comparing the supergravity constraints with Maurer Cartan equations (2.3) we have the following translation

$$f^a_{\alpha\beta} \to -\mathcal{T}^a_{\alpha\beta}\,, \tag{5.9}$$
$$f^{(ab)}_{\alpha\beta} \to -\mathcal{R}^{ab}_{\alpha\beta}\,, \tag{5.10}$$
$$f^{(ab)}_{cd} \to -R_{cd}{}^{ab}\,, \tag{5.11}$$
$$f^\alpha_{a\beta} \to -\mathcal{T}^\alpha_{a\beta}\,, \tag{5.12}$$
$$f^\alpha_{(ab)\beta} \to \frac{1}{4}(\Gamma_{ab})^\alpha{}_\beta \tag{5.13}$$
$$f^a_{(cd)b} \to -\eta_{cb}\delta^a_b\,, \tag{5.14}$$
$$f^{(ab)}_{(el)\,(cd)} \to \delta^a_e \eta_{lc} \delta^b_d\,. \tag{5.15}$$

Thus we may rewrite (5.6) as Maurer Cartan equations with soft structure "constants" $f^\Lambda_{\Sigma\Delta}(\Phi^\alpha)$ depending on some covariant superfields Φ^α[5], whose properties are defined in the Appendix A.2.

$$
\left(dL^\Lambda + \frac{1}{2}L^\Delta \wedge L^\Sigma f_{\Sigma\Delta}{}^\Lambda(\Phi)\right)\mathbf{T}_\Lambda = 0\,. \tag{5.16}
$$

[5]These superfields are the forms, the dilaton and dilatino of the background. As the dilaton is a scalar this requires it to be constant, which occurs for the $D3$-brane solution and therefore for $D3$ the superspace is a coset superspace.

This is the statement that there exists a nilpotent differential operator

$$\mathcal{D} = d - L^\Lambda T_\Lambda \qquad \mathcal{D}^2 = 0 \tag{5.17}$$

and the generators of the soft algebra satisfy $[\,\mathbf{T}_\Lambda, \mathbf{T}_\Sigma\,\} = f_{\Lambda\Sigma}{}^\Delta(\Phi^\alpha)\mathbf{T}_\Delta$. This is only possible iff

$$\mathcal{D} f_{\Sigma\Delta}{}^\Lambda(\Phi^\alpha) \equiv d f_{\Sigma\Delta}{}^\Lambda - \Phi^\beta (L^\Pi \mathbf{T}_\Pi)_\beta{}^\alpha \partial_\alpha f^\Lambda_{\Sigma\Delta} = 0 \tag{5.18}$$

since $\mathcal{D}^2 f_{\Sigma\Delta}{}^\Lambda(\Phi^\alpha) = 0$. This is in fact equivalent to the statement that the structure "constants" of the soft algebra have to satisfy the following generalized Jacobi identities:

$$0 = \Sigma_3^\Sigma \Sigma_2^\Delta \left(f_{\Delta\Sigma}{}^\Pi \Sigma_1^\Gamma f_{\Gamma\Pi}{}^\Lambda - \Phi^\beta (\Sigma_1^\Pi \mathbf{T}_\Pi)_\beta{}^\alpha \partial_\alpha f_{\Delta\Sigma}{}^\Lambda \right) + \text{cyclic}(1 \to 2 \to 3)\,. \tag{5.19}$$

Now we may use the dictionary (5.9)–(5.15) to identify the soft algebra with structure functions depending on curvatures and torsions of supergravity. There is the universal part containing M, which will be H

$$\begin{aligned}
[M_{ab}, M_{cd}] &= \eta_{a[c} M_{d]b} - \eta_{b[c} M_{d]b}\,, \\
[P_a, M_{bc}] &= \eta_{a[b} P_{c]}\,, \qquad [M_{ab}, Q_\alpha] = -\frac{1}{4}(\Gamma_{ab} Q)_\alpha
\end{aligned} \tag{5.20}$$

and

$$[P_a, P_b] = -\mathcal{R}_{ab}{}^{cd} M_{cd}\,, \tag{5.21}$$

$$\begin{aligned}
[P_a, Q_\alpha] &= -\mathcal{T}_{a\alpha}{}^\beta Q_\beta\,, \\
\{Q_\alpha, Q_\beta\} &= -\mathcal{T}_{\alpha\beta}{}^a P_a - \mathcal{R}_{\alpha\beta}{}^{ab} M_{ab}\,.
\end{aligned} \tag{5.22}$$

One should read these formulas carefully. In the case that one has a product geometry as is $AdS \times S$ the curvature $\mathcal{R}_{ab}{}^{cd}$ will split into two parts and therefore the generators M_{ab} which are off-diagonal, i.e. they have an index in the tangent space of each of the two factors in the product space, will not appear on the r.h.s. of (5.21). For consistency of the algebra therefore these generators should also be dropped in (5.20). An example of this is that the tangent space group of $AdS_{p+2} \times S^{d-p-2}$ is $\mathrm{SO}(p+1,1) \times \mathrm{SO}(d-p-2)$ in stead of the full $\mathrm{SO}(d,1)$ as in flat space.

For flat space we have

$$\mathcal{T}^a_{\alpha\beta} = (\Gamma^a)_{\alpha\beta}\,, \qquad \mathcal{T}^\beta_{\alpha a} = \mathcal{R} = 0\,. \tag{5.23}$$

For the general case the solution is given by

$$E^\alpha = \left(\frac{\sinh \mathcal{M}}{\mathcal{M}} \right)^\alpha_{\ \beta} (D\Theta)^\beta\,, \tag{5.24}$$

$$E^a = e^a - \Theta^\alpha \mathcal{T}^a_{\alpha\beta} \left(\frac{\sinh \mathcal{M}/2}{\mathcal{M}^2} \right)^\beta_{\ \gamma} (D\Theta)^\gamma\,, \tag{5.25}$$

$$\Omega^{ab} = \omega^{ab} - \Theta^\alpha \mathcal{R}^{ab}_{\alpha\beta} \left(\frac{\sinh \mathcal{M}/2}{\mathcal{M}^2}\right)^\beta_\gamma (D\Theta)^\gamma , \tag{5.26}$$

$$(\mathcal{M}^2)^\alpha_\beta = -\frac{1}{4}(\gamma_{ab})^\alpha{}_\gamma \Theta^\gamma \Theta^\delta \mathcal{R}^{(ab)}_{\delta\beta} - \mathcal{T}^\alpha_{a\gamma}\Theta^\gamma\Theta^\delta\mathcal{T}^a_{\delta\beta} . \tag{5.27}$$

One can verify that these results in terms of the structure functions, using the dictionary (5.9)–(5.15) are the same as the one in the supercoset space We find the superisometries the same way as before and therefore the result is given (4.43)–(4.45) where the dictionary (5.9)–(5.15) has to be used to replace the expressions for constant $f^\Lambda_{\Sigma\Delta}$ by their supergravity counterparts which are covariantly constant.

6. Discussion

Thus we have found the superisometries, the combination of the transformations of coordinates (X, θ) of the near-horizon superspace and the compensating Lorentz transformations, under which the vielbeins and the spin connection of the superspace are invariant. In the picture of the product space $AdS_{p+2} \times S^{d-p-2}$ where a $\frac{G}{H}$ supercoset construction is available [5], the isometries are given in (4.43)–(4.45). The result covers all cases of the superconformal algebras **G**. The data required to construct the whole superspace and its isometries consists of the the algebra and the Killing spinors and vectors at $\theta = 0$, which have been discussed in sec. 4.2. Alternatively one can start from supergravity, i.e. one takes a solution to the supergravity field equations (with vanishing fermions) and the Killing vectors and Killing spinors and compensating Lorentz transformation can be obtained as the transformations that leave the solution invariant. To construct the higher order θ components one identifies the supertorsion and curvature components with the structure 'constants' of the algebra, the dictionary has been given in (5.9)–(5.15), and we can apply our general formula in terms of structure constants. It has been observed that for the maximally supersymmetric solutions with vanishing fermions the supertorsions and curvatures are only covariantly constant [7]. This happens for instance if the near-horizon geometry is given in cartesian coordinates. Fortunately our construction of the superisometries thus not demand structure constants, but only covariantly constant structure functions. The derivation of this result can be found in sec. 5.

Having established the full set of isometries in AdS superspace, we may use them for few applications, in particular we would like to study the condition of the BPS states on the brane in an AdS background.

1. *BPS States of Branes in AdS.*
   The classical worldvolume action has two types of fermionic symmetries: a global one, the fermionic isometry of the flat superspace and a local one, the so-called κ-symmetry. On the fermionic variables θ they act as

$$\delta_f \theta = -\epsilon + (1 + \Gamma)\kappa . \tag{6.1}$$

Here Γ defines the κ-symmetry depends in general on (X, θ) and on the worldvolume forms F, i.e. we have $\Gamma(X, \Theta, F)$. The matrix Γ satisfies $\Gamma^2 = 1$ and $\mathrm{tr}\,\Gamma = 0$. To identify the equation which defines some configuration to be a BPS configuration of the brane in flat space, one has to require that the total transformation of the fermionic variables vanishes at vanishing fermions

$$\delta_{BPS}\theta \equiv (\delta\theta)_{\theta=0} = (-\epsilon + (1+\Gamma)\kappa)_{\theta=0} = 0\,. \tag{6.2}$$

Now we may observe that for vanishing fermions

$$\delta_{BPS}(1-\Gamma)\theta = -(1-\Gamma)_{\theta=0}\epsilon = 0 \tag{6.3}$$

due to the fact that $(1-\Gamma)(1+\Gamma) = 0$. This form for the preservation of supersymmetry was first discussed in [20] and was subsequently considered in [21, 22, 23, 24]. The number of independent zero modes ϵ of this equation can be 1/2 or less than the total number of components of fermions θ.

What from this analysis carries over to the case when the branes are in AdS superspace? One has to start with the κ-symmetric brane actions in the AdS-superspace background and formulate the total supersymmetry transformation on the fermions. It consists of the fermionic superisometry of the background and of a local κ-symmetry.

$$(\delta\theta^{\dot\alpha})E_{\dot\alpha}{}^{\alpha}(X,\theta) = (\delta_{AdS}\theta^{\dot\alpha})E_{\dot\alpha}{}^{\alpha}(X,\theta) + (1+\Gamma)\kappa\,. \tag{6.4}$$

Here we have taken into account that in the Killing spinor gauge the gravitino superfield vanishes, $E^{\alpha}_{\mu}(X,\Theta) = 0$. The condition of unbroken supersymmetry of the BPS state on the brane at $\theta = \delta\theta = 0$ is reduced to the condition

$$\Sigma^{\alpha}(X, \theta = 0) + (1 + \Gamma(X, \theta = 0, F))\kappa = 0\,. \tag{6.5}$$

One can as before multiply this condition on $(1 - \Gamma(X, \theta = 0, F))$ and get[6]

$$(1 - \Gamma(X,F))^{\alpha}_{\ \beta}\,\Sigma^{\beta}_{0}(X) = 0\,. \tag{6.6}$$

Here $\Sigma^{\beta}_{0}(X)$ as it follows from our complete set of transformations, is the Killing spinor of the background. Note also that κ-symmetry transformation carries the information about the brane action whose symmetries are investigated as well as the information on the background. In particular one can look at any D-p-brane of IIB theory in $AdS_5 \times S^5$ background and find all possible BPS states. For example one can study a D5 brane in the near horizon background of D3 brane. The issue of BPS states of such configuration was studied in [26] following [27]. The derivation of the BPS condition there is surprisingly complicated and never

[6]This condition was also suggested in [25]

actually using the direct supersymmetry of the D5 brane in $AdS_5 \times S^5$ background which is a combination of the background isometries and κ-symmetry. Thus we would like to stress here that (6.6) may have many possible solutions, only small part of which was recently studied in [25].

2. *Use of superisometries for GS Superstring in AdS background*
A classical GS Superstring in the near horizon superspace of the D3 branes has the combinations of symmetries as shown in (6.4).

$$(\delta\theta^{\dot\alpha})E_{\dot\alpha}{}^{\alpha}(X,\theta) = (\delta_{AdS^5\times S^5}\theta^{\dot\alpha})E_{\dot\alpha}{}^{\alpha}(X,\theta) + (1+\Gamma_{string})\kappa \tag{6.7}$$

supplemented with the corresponding transformations of the bosonic fields. Here the superspace vielbeins $E_{\dot\alpha}{}^{\alpha}(X,\theta)$ are defined for the near horizon D3 geometry, and the generator of a local κ-symmetry is the one for the string in the D3 background. The expression for the matrix $\mathcal{M}^2$ simplifies in any of the gauges $\Theta_+ = 0$ or $\Theta_- = 0$, it becomes a nilpotent matrix and therefore squares to zero. An easy way to see this is to observe that

$$\mathcal{M}^4 A = [\Theta[\Theta[\Theta[\Theta, A]]]]_{\Theta_\pm=0} = 0 \tag{6.8}$$

and therefore

$$(\mathcal{M}^4 A)_{\Theta_\pm=0} = 0 \tag{6.9}$$

For $\Theta_+ = 0$ or $\Theta_- = 0$ the 4 objects with Θ accumulate the scaling weight $4\times\pm\frac{1}{2}$ and therefore together with the scaling weight of the fermionic operator in A the weight of the operator in this multiple commutator becomes equal to $\pm 2 \pm \frac{1}{2}$, i.e. $\pm\frac{5}{2}$. Such operators do not exist in our algebras and therefore this expression vanishes for arbitrary A. Note that in case $\Theta_- = 0$ a stronger restriction is available. In addition to $\Theta_- = 0$ one can show that also $(D\Theta)_- = 0$ and in such case even $\mathcal{M}^2 D\Theta = 0$. Indeed here we use

$$\mathcal{M}^2 D\Theta = [\Theta[\Theta, D\Theta] \tag{6.10}$$

and therefore

$$(\mathcal{M}^2 D\Theta)_{\Theta_-=0} = 0 \tag{6.11}$$

as the weight of all 3 fermionic operators sums up to 3/2. Here again we take into account that the fermion operators with such weight are not available in our algebras. This leads to the following simplification of the isometries in the gauge-fixed form of the action. In particular, if we choose, as in [9], the gauge $\Theta_- = \theta_- = 0$, we find that the contribution from isometries to the total transformations is simplified:

$$-\delta\theta_- = \epsilon_{0-} + \Delta(\kappa) = 0 \tag{6.12}$$

and

$$-\delta\theta_+ = \epsilon_{0+} + \frac{1}{2}(\mathcal{K}^{-1}\mathcal{M}^2\mathcal{K})_{+-}\epsilon_{0-} + \Delta(\kappa) \tag{6.13}$$

where $\epsilon_{\pm 0}$ are constant spinors and $\Delta(\kappa)$ is the contribution from the local κ-symmetry. The second term in $\delta\theta_+$ is only quadratic in fermionic variables. These equations give the basis for the derivation of the symmetries of the gauge-fixed IIB string action in AdS background [9]. From these symmetries one should be able to derive the Ward Identities which will be responsible for the properties of the loop corrections. Since the rigid symmetries of the gauge-fixed action will form the superconformal algebra, the spectrum of states must also form a representations of this algebra. This symmetry may help us to find the spectrum of states of the string in AdS.

3. *Boundary limit*
   One may try to use the superconformal symmetry of the near horizon superspace to find the limit to the boundary of the AdS space, as suggested to us by S. Shenker. This will provide a particular realization of the superconformal algebras closely related to the properties of the conformal field theories at the boundary. It may also help to clarify the AdS/CFT correspondence conjectured by Maldacena. This project is currently under investigation in [28]. One of the striking new results in this study is the derivation of the off-shell harmonic superspace of super Yang-Mills theory from the boundary limit of the supergravity/IIB string compactified on $AdS_5 \times S^5$.

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A. Conventions

A.1 Super differential forms

To define differential forms we use the basic rule

$$dZ^M \wedge dZ^N = -(-)^{MN} dZ^N \wedge dZ^M \,, \tag{A.1}$$

where $(-)^{MN} = 1$ unless both indices are spinorial. The components of an n-form Φ_n are defined through

$$\Phi_n = \frac{1}{n!} dZ^{M_1} \wedge \ldots \wedge dZ^{M_p} \Phi_{M_p \ldots M_1} \,. \tag{A.2}$$

The exterior derivative d is given by

$$d\Phi_n = \frac{1}{n!} dZ^{M_1} \wedge \ldots \wedge dZ^{M_p} \wedge dZ^N \partial_N \phi_{M_p \ldots M_1}\,, \tag{A.3}$$

and is nilpotent

$$d^2 = 0\,. \tag{A.4}$$

d starts from the right which means that

$$d(\Phi_m \wedge \Phi_n) = \Phi_m \wedge (d\Phi_n) + (-)^n (d\Phi_m) \wedge \Phi_n\,. \tag{A.5}$$

Also we take the convention that d commutes with θ^A.

A useful second operator is the Lie derivative along a vector field $\Xi = \Xi^M \frac{\partial}{\partial Z^M}$ of an n-form Φ_n

$$\mathcal{L}_\Xi \Phi_n = \frac{1}{n!} dZ^{M_1} \wedge \ldots \wedge dZ^{M_n} \left(\Xi^N \partial_N \Phi_{M_n \ldots M_1} + n(\partial_{M_n} \Xi^N) \Phi_{N M_{(n-1)} \ldots M_1} \right)\,. \tag{A.6}$$

A.2 Gauging (soft) superalgebras

We consider a supergroup G with associated superalgebra $\mathbf{G}$.[7] Consider a bunch of 'covariant' fields Φ^α, they transform in some representation labeled by α of G. The generators of the algebra are $\mathbf{T}_\Lambda$ and satisfy

$$[\,\mathbf{T}_\Lambda, \mathbf{T}_\Sigma\,\} \equiv \mathbf{T}_\Lambda \mathbf{T}_\Sigma \mp \mathbf{T}_\Sigma \mathbf{T}_\Lambda = f_{\Lambda\Sigma}{}^\Delta (\Phi^\alpha) \mathbf{T}_\Delta\,, \tag{A.7}$$

where the + sign is taken if both Λ and Σ are fermionic indices. We have included the possibility of *soft* algebras, where the structure 'constants' are actually structure functions.

We define a $\mathbf{G}$ valued object A as

$$A = A^\Lambda \mathbf{T}_\Sigma\,. \tag{A.8}$$

It follows that for two bosonic $\mathbf{G}$ valued objects A and B

$$[\,A, B\,] = [\,A^\Lambda \mathbf{T}_\Lambda, B^\Sigma \mathbf{T}_\Sigma\,] = B^\Sigma A^\Lambda [\,\mathbf{T}_\Lambda, \mathbf{T}_\Sigma\,\} = B^\Sigma A^\Lambda f_{\Lambda\Sigma}{}^\Delta \mathbf{T}_\Delta\,. \tag{A.9}$$

We will consider the $\mathbf{T}$ as "active" operators. Take, e.g. on the fields field Φ^α an infinitesimal group transformation is denoted by

$$\delta(\Sigma)\Phi^\alpha = \Phi^\beta \Sigma^\Lambda \mathbf{T}_{\boldsymbol{\Lambda}\beta}{}^\alpha \equiv \Phi^\beta \Sigma_\alpha{}^\beta\,, \qquad \text{where} \qquad \Sigma = \Sigma^\Lambda \mathbf{T}_\Lambda\,. \tag{A.10}$$

The infinitesimal parameters Σ are $\mathbf{G}$ valued objects. If we apply a second variation it acts as

$$\delta(\Sigma_1)\delta(\Sigma_2)\Phi^\alpha = \Phi^\gamma \Sigma_{2\gamma}{}^\beta \Sigma_{1\beta}{}^\alpha\,, \tag{A.11}$$

thus T_{Λ_1} works on (ΦT_{Λ_2}).

[7] This section is based on [11] but written in form notation.

Now we can look at the commutator of two such transformations on a covariant field. It follows from (A.7) that they should satisfy

$$[\delta(\Sigma_1), \delta(\Sigma_2)] = \delta(\Sigma_2^\Delta \Sigma_1^\Pi f_{\Pi\Delta}{}^\Lambda)\,. \tag{A.12}$$

We have to take into account that the structure constants also transform

$$\delta(\Sigma) f_{\Sigma\Delta}{}^\Lambda = \Phi^\beta \Sigma_\beta{}^\alpha \frac{\partial}{\partial \Phi^\alpha} f_{\Sigma\Delta}{}^\Lambda\,. \tag{A.13}$$

The Jacobi identities which follow from $0 = [\delta(\Sigma_1), [\delta(\Sigma_2), \delta(\Sigma_3)] + \text{cyclic}(1 \to 2 \to 3)$ are then

$$0 = \Sigma_3^\Sigma \Sigma_2^\Delta \left(f_{\Delta\Sigma}{}^\Pi \Sigma_1^\Gamma f_{\Gamma\Pi}{}^\Lambda - \delta(\Sigma_1) f_{\Delta\Sigma}{}^\Lambda \right) + \text{cyclic}(1 \to 2 \to 3)\,. \tag{A.14}$$

We want to gauge the superalgebra $\mathbf{G}$. The parameters Σ are now coordinate dependent $\Sigma(Z)$. The gauge field A is a $\mathbf{G}$ valued 1-form. We introduce a covariant derivative denoted by

$$\mathcal{D} = d - A\,. \tag{A.15}$$

On a G-covariant field it acts as

$$\mathcal{D}\Phi^\alpha = d\Phi^\alpha - \Phi^\beta A_\beta{}^\alpha\,. \tag{A.16}$$

It acts *from the right*, consistent with the convention that d acts from the right. We define the transformation of the gauge field A, such that in $\delta(\Sigma)(\mathcal{D}\Phi)^\alpha$ there is no derivative on the parameters Σ^Λ. This can be achieved by taking

$$\begin{aligned} \delta(\Sigma) A &= d\Sigma + [\,A, \Sigma\,] \\ &= \left(d\Sigma^\Lambda + \Sigma^\Delta A^\Sigma f_{\Sigma\Delta}{}^\Lambda \right) \mathbf{T}_\Lambda\,. \end{aligned} \tag{A.17}$$

Trying to close the commutator (A.12) on A, we find that

$$\mathcal{D} f_{\Sigma\Delta}{}^\Lambda = d f_{\Sigma\Delta}{}^\Lambda - \Phi^\beta (A^\Pi \mathbf{T}_\Pi)_\beta{}^\alpha \partial_\alpha f_{\Sigma\Delta}^\Lambda \tag{A.18}$$

Therefore one can gauge the *soft* algebras if the structure functions are covariantly constant.

One can construct the G-covariant curvature F, which is a $\mathbf{G}$ valued 2 form, by

$$\mathcal{D}^2 = -F \tag{A.19}$$

and it follows that

$$\begin{aligned} F &= dA - A \wedge A \\ &= \left(dA^\Lambda + \frac{1}{2} A^\Delta \wedge A^\Sigma f_{\Sigma\Delta}{}^\Lambda \right) \mathbf{T}_\Lambda\,. \end{aligned} \tag{A.20}$$

F transforms in the adjoint (only if (A.18) is satisfied), i.e.

$$\delta(\Sigma)F = [F, \Sigma] = -F^{\Sigma}\Sigma^{\Delta} f_{\Delta\Sigma}{}^{\Lambda}\mathbf{T}_{\Lambda} \,. \tag{A.21}$$

The adjoint representation matrices are given by

$$(\mathbf{T_\Lambda})_{\Sigma}{}^{\Delta} = -f_{\Lambda\Sigma}{}^{\Delta} \tag{A.22}$$

and satisfy (2.1) by virtue of the Jacobi identities. The curvatures satisfy the Bianchi identities (again only if (A.18) is satisfied)

$$0 = \mathcal{D}F = dF + [A, F] \,, \tag{A.23}$$

using (A.16).

B. Coset manifolds and superisometries

Consider arbitrary elements g and h of the groups G and H respectively. We define equivalence classes in G: two elements g and g' belong to the same equivalence class iff they can be connected by a right multiplication with an element of H, i.e.

$$g = g'\, h \,. \tag{B.1}$$

This equivalence class is called the left coset of g. The set of all cosets is the coset manifold denoted by G/H.

Now we can characterize each coset by a coset representative $\mathcal{G}(Z)$, labelled by as many coordinates Z as we need, typically for the supercoset spaces we consider $Z = \{X^{\mu}, \theta^{\dot\alpha}\}$. It parametrizes the coset manifold if each coset contains exactly one of the $\mathcal{G}(Z)$'s. Once we have chosen a representative it is clear that every group element g can be decomposed into

$$g = \mathcal{G}(Z)h \,, \tag{B.2}$$

where $\mathcal{G}(Z)$ is the representative of the coset to which g belongs and h acts in this coset.

Now it is clear that a product with an arbitrary group element g of G with a coset representative $\mathcal{G}(Z)$ can bring you to another coset and

$$g\mathcal{G}(Z) = \mathcal{G}(Z')h \,, \tag{B.3}$$

where in general

$$Z' = Z'(g, Z) \,, \qquad h = h(g, Z) \,. \tag{B.4}$$

The *global* group transformations g correspond to the (super)isometry group of the (super)coset space.

We define the left-invariant Cartan 1-forms

$$\mathcal{G}(Z)^{-1}d\mathcal{G}(Z) = L(Z)\,. \tag{B.5}$$

Since $L(Z)$ is a group element close to the identity it is a $\mathbf{G}$ valued super 1-form

$$L = L^\Lambda \mathbf{T}_\Lambda = dZ^M L_M{}^\Lambda \mathbf{T}_\Lambda\,. \tag{B.6}$$

They are invariant under a *global* G-transformation from the left

$$\mathcal{G}(Z) \to g\mathcal{G}(Z)\,. \tag{B.7}$$

Indeed, we have

$$(g\mathcal{G}(Z))^{-1}d(g\mathcal{G}(Z)) - \mathcal{G}(Z)^{-1}d\mathcal{G}(Z) = \mathcal{G}(Z)^{-1}g^{-1}gd\mathcal{G}(Z) - \mathcal{G}(Z)^{-1}d\mathcal{G}(Z) = 0\,. \tag{B.8}$$

The Cartan 1-forms satisfy the Maurer Cartan equations (2.3) which is easily deduced by substituting $L = \mathcal{G}^{-1}d\mathcal{G}$ into this equation.

We can rewrite (B.8) by using (B.3) and obtain

$$0 = h^{-1}L(Z')h + h^{-1}dh - L(Z)\,. \tag{B.9}$$

Now we are interested in infinitesimal transformations of the coordinates

$$Z'^M = Z^M - \Xi^M(Z)\,. \tag{B.10}$$

The compensating H-transformation are infinitesimal and we define

$$h = 1 - \Lambda = 1 - \Lambda^i \mathbf{H}_i\,. \tag{B.11}$$

One easily derives that

$$L(Z') = L(Z) - \mathcal{L}_\Xi L(Z)\,. \tag{B.12}$$

Therefore (B.9) becomes

$$\begin{aligned} 0 &= \mathcal{L}_\Xi L + d\Lambda + [\,L(Z), \Lambda\,] \\ &= \left(\mathcal{L}_\Xi L^\Lambda + d\Lambda^i \delta_i{}^\Lambda + \Lambda^i L^\Sigma f_{\Sigma i}{}^\Lambda\right) \mathbf{T}_\Lambda\,. \end{aligned} \tag{B.13}$$

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